

SUBDIFFERENTIAL TEST FOR OPTIMALITY

Florence JULES and Marc LASSONDE
 LAMIA, Université des Antilles et de la Guyane
 97159 Pointe à Pitre, France

E-mail: florence.jules@univ-ag.fr, marc.lassonde@univ-ag.fr

Abstract. We provide a first-order necessary and sufficient condition for optimality of lower semicontinuous functions on Banach spaces using the concept of subdifferential. From the sufficient condition we derive that any subdifferential operator is monotone absorbing, hence maximal monotone when the function is convex.

Keywords: lower semicontinuity, subdifferential, directional derivative, first-order condition, optimality criterion, maximal monotonicity.

2010 Mathematics Subject Classification: 49J52, 49K27, 26D10, 26B25.

1 Introduction

First-order sufficient conditions for optimality in terms of derivatives or directional derivatives are well known. Typical such conditions are variants of Minty variational inequalities. Let us quote for example the following simple result (see [3]): *Let f be a real-valued function which is continuous on a neighbourhood D centred at a in \mathbb{R}^n and differentiable on $D \setminus \{a\}$. Then, f has a local minimum value at a if $(x - a) \cdot \nabla f(x) > 0$ for all $x \in D \setminus \{a\}$.* For first-order conditions in terms of directional derivatives, we refer to [5].

The main objective of this note is to establish a necessary and sufficient condition for optimality of nonsmooth lower semicontinuous functions on Banach spaces using subdifferentials. To this end, we first provide a link between the directional derivative of a function and its dual companion represented by the subdifferential (Theorem 2.1). Then, we prove a sharp version of the mean value inequality for lower semicontinuous function using directional derivative (Lemma 3.1). Finally, we combine both results to obtain our subdifferential test for optimality (Theorem 3.3). A discussion on the sufficient condition follows where it is shown that any subdifferential operator is *monotone absorbing*, a property which reduces to maximal monotonicity when the function is convex (Theorem 3.4).

This paper complements our previous work [8] which concerned only the elementary subdifferentials.

2 Directional Derivative and Subdifferential

In the sequel, X is a real Banach space with unit ball B_X , X^* is its topological dual with unit ball B_{X^*} , and $\langle \cdot, \cdot \rangle$ is the duality pairing. For a subset $A \subset X$, $x \in X$ and $\lambda > 0$, we let $d_A(x) := \inf_{y \in A} \|x - y\|$ and $B_\lambda(A) := \{y \in X : d_A(y) \leq \lambda\}$. All the functions $f : X \rightarrow]-\infty, +\infty]$ are assumed to be lower semicontinuous and *proper*, which means that the set $\text{dom } f := \{x \in X : f(x) < \infty\}$ is nonempty. The (radial or lower Dini) *directional derivative* of a function f at a point $\bar{x} \in \text{dom } f$ is given by:

$$\forall d \in X, \quad f'(\bar{x}; d) := \liminf_{t \searrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t}.$$

A *subdifferential* of a lower semicontinuous function f is a set-valued operator $\partial f : X \rightrightarrows X^*$ which coincides with the usual convex subdifferential whenever f is convex, that is,

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y - \bar{x} \rangle + f(\bar{x}) \leq f(y), \forall y \in X\},$$

and satisfies elementary stability properties like $\partial(f - x^*)(x) = \partial f(x) - x^*$ for every $x \in X$ and $x^* \in X^*$. In this work, we also require that the subdifferentials satisfy the following basic calculus rule:

Separation Principle. For any lower semicontinuous f, φ with φ convex Lipschitz near $\bar{x} \in \text{dom } f \cap \text{dom } \varphi$, if $f + \varphi$ admits a local minimum at \bar{x} , then $0 \in \partial f(\bar{x}) + \partial \varphi(\bar{x})$.

Examples. The Clarke subdifferential [4], the Michel-Penot subdifferential [9], the Ioffe A-subdifferential [7] satisfy the Separation Principle in any Banach space. The limiting versions of the elementary subdifferentials (proximal, Fréchet, Hadamard, Gâteaux, ...) satisfy the Separation Principle in appropriate Banach spaces (see, e.g., [8, 2, 10] and the references therein).

The link between the directional derivative and the subdifferential is described via the following ε -enlargement of the subdifferential:

$$\check{\partial}_\varepsilon f(\bar{x}) := \{x_\varepsilon^* \in X^* : x_\varepsilon^* \in \partial f(x_\varepsilon) \text{ with } \|x_\varepsilon - \bar{x}\| \leq \varepsilon, |f(x_\varepsilon) - f(\bar{x})| \leq \varepsilon, \langle x_\varepsilon^*, x_\varepsilon - \bar{x} \rangle \leq \varepsilon\}.$$

Theorem 2.1 *Let X be a Banach space, $f : X \rightarrow]-\infty, +\infty]$ be lower semicontinuous, $\bar{x} \in \text{dom } f$ and $d \in X$. Then, for every $\varepsilon > 0$, the sets $\check{\partial}_\varepsilon f(\bar{x})$ are nonempty and*

$$f'(\bar{x}; d) \leq \inf_{\varepsilon > 0} \sup \langle \check{\partial}_\varepsilon f(\bar{x}), d \rangle. \quad (1)$$

Proof. We first show that the sets $\check{\partial}_\varepsilon f(\bar{x})$ are nonempty. The arguments are standard, we give them for completeness. Since f is lower semicontinuous at \bar{x} , there is $\lambda \in]0, \varepsilon[$ such that

$$f(\bar{x}) \leq \inf f(\bar{x} + \lambda B_X) + \varepsilon. \quad (2)$$

Applying Ekeland's variational principle [6], we find $x_\varepsilon \in X$ such that

$$\|x_\varepsilon - \bar{x}\| < \lambda, \quad f(x_\varepsilon) \leq f(\bar{x}), \quad \text{and} \quad (3a)$$

$$y \mapsto f(y) + (\varepsilon/\lambda)\|y - x_\varepsilon\| \text{ admits a local minimum at } x_\varepsilon. \quad (3b)$$

Thanks to (3b), we may invoke the Separation Principle at point x_ε with the convex Lipschitz function $\varphi : y \mapsto (\varepsilon/\lambda)\|y - x_\varepsilon\|$ to obtain a subgradient $x_\varepsilon^* \in \partial f(x_\varepsilon)$ such that $-x_\varepsilon^* \in \partial \varphi(x_\varepsilon)$. From (3a) and (2), we derive that $\|x_\varepsilon - \bar{x}\| \leq \varepsilon$ and $|f(x_\varepsilon) - f(\bar{x})| \leq \varepsilon$, while from $-x_\varepsilon^* \in \partial \varphi(x_\varepsilon)$ we get $\|x_\varepsilon^*\| \leq \varepsilon/\lambda$, so combining with (3a) we find $\langle x_\varepsilon^*, x_\varepsilon - \bar{x} \rangle \leq (\varepsilon/\lambda)\lambda = \varepsilon$. This completes the proof of the nonemptiness of $\check{\partial}_\varepsilon f(\bar{x})$.

We now proceed to the proof of Formula (1). Let $d \neq 0$ and let $\lambda < f'(\bar{x}; d)$. It suffices to show that there exists a sequence $((x_n, x_n^*))_n \subset \partial f$ verifying

$$x_n \rightarrow \bar{x}, \quad f(x_n) \rightarrow f(\bar{x}), \quad \limsup_n \langle x_n^*, x_n - \bar{x} \rangle \leq 0, \quad \text{and} \quad (4a)$$

$$\lambda \leq \liminf_n \langle x_n^*, d \rangle. \quad (4b)$$

Let $t_0 \in]0, 1]$ such that

$$0 \leq f(\bar{x} + td) - f(\bar{x}) - \lambda t, \quad \forall t \in [0, t_0] \quad (5)$$

and let $z^* \in X^*$ such that

$$\langle z^*, td \rangle = -\lambda t, \quad \forall t \in \mathbb{R} \quad \text{and} \quad \|z^*\| = |\lambda|/\|d\|. \quad (6)$$

Set $K := [\bar{x}, \bar{x} + t_0 d]$ and $g := f + z^*$. Then, (5) can be rewritten as

$$g(\bar{x}) \leq g(x), \quad \forall x \in K.$$

Let $\delta > 0$ such that g is bounded from below on $B_\delta(K)$. For each positive integer n such that $1/n < \delta$, let $r_n > 0$ such that

$$g(\bar{x}) - 1/n^2 < \inf_{B_{r_n}(K)} g$$

and let then $\alpha_n > 0$ such that

$$\inf_{B_{r_n}(K)} g \leq \inf_{B_\delta(K)} g + \alpha_n r_n.$$

It readily follows from these inequalities that

$$g(\bar{x}) \leq g(x) + \alpha_n d_K(x) + 1/n^2, \quad \forall x \in B_\delta(K). \quad (7)$$

Applying Ekeland's variational principle to the function $x \mapsto g(x) + \alpha_n d_K(x)$, we obtain a sequence $(x_n)_n \subset B_\delta(K)$ such that

$$\|\bar{x} - x_n\| < 1/n, \quad (8a)$$

$$g(x_n) + \alpha_n d_K(x_n) \leq g(\bar{x}) \quad (8b)$$

$$x \mapsto g(x) + \alpha_n d_K(x) + (1/n)\|x - x_n\| \text{ admits a local minimum at } x_n. \quad (8c)$$

In view of (8c), we may apply the Separation Principle with the given f and $\varphi = z^* + \alpha_n d_K + (1/n)\|\cdot - x_n\|$ to obtain $x_n^* \in \partial f(x_n)$, $\zeta_n^* \in \partial d_K(x_n)$ and $\xi^* \in B_{X^*}$ such that

$$x_n^* = -z^* - \alpha_n \zeta_n^* - (1/n)\xi^*. \quad (9)$$

We show that the sequence $((x_n, x_n^*))_n \subset \partial f$ satisfies (4a) and (4b).

Proof of (4a). Combining (6) and (8b) we get

$$f(x_n) \leq f(\bar{x}) + \langle z^*, \bar{x} - x_n \rangle \leq f(\bar{x}) + (|\lambda|/\|d\|)\|\bar{x} - x_n\|. \quad (10)$$

Since f is lower semicontinuous, (8a) and (10) show that $x_n \rightarrow \bar{x}$ and $f(x_n) \rightarrow f(\bar{x})$. On the other hand, since $\langle \zeta_n^*, x - x_n \rangle \leq d_K(x) - d_K(x_n) \leq 0$ for all $x \in K$, it follows from (6) and (9) that

$$\langle x_n^*, \bar{x} - x_n \rangle \geq \langle z^*, x_n - \bar{x} \rangle - (1/n)\langle \xi^*, \bar{x} - x_n \rangle \geq -(|\lambda|/\|d\|)\|x_n - \bar{x}\| - (1/n)\|x_n - \bar{x}\|,$$

showing that $\limsup_n \langle x_n^*, x_n - \bar{x} \rangle \leq 0$.

Proof of (4b). From (9) we derive

$$\langle x_n^*, d \rangle = \langle -z^*, d \rangle - (1/n)\langle \zeta_n^*, d \rangle - \alpha_n \langle \zeta_n^*, d \rangle \geq \lambda - (1/n)\|d\| - \alpha_n \langle \zeta_n^*, d \rangle. \quad (11)$$

We claim that $\langle \zeta_n^*, d \rangle \leq 0$. Indeed, let $P_K x_n \in K$ such that $\|x_n - P_K x_n\| = d_K(x_n)$. We have $P_K x_n \neq \bar{x} + t_0 d$ for large n since $x_n \rightarrow \bar{x}$, so there exists $t_n > 0$ such that $t_0 d = \bar{x} + t_0 d - \bar{x} = t_n(\bar{x} + t_0 d - P_K x_n)$. It follows that

$$\begin{aligned} (t_0/t_n)\langle \zeta_n^*, d \rangle &= \langle \zeta_n^*, \bar{x} + t_0 d - P_K x_n \rangle = \langle \zeta_n^*, \bar{x} + t_0 d - x_n \rangle + \langle \zeta_n^*, x_n - P_K x_n \rangle \\ &\leq -d_K(x_n) + \|x_n - P_K x_n\| = 0. \end{aligned}$$

Hence $\langle \zeta_n^*, d \rangle \leq 0$. We therefore conclude from (11) that $\lambda \leq \liminf_n \langle x_n^*, d \rangle$. This completes the proof. \blacksquare

Remarks. 1. For convex functions, the inequality in (1) becomes an equality, and the formula is due to Taylor [11] and Borwein [1].

2. For elementary subdifferentials, Formula (1) was proved to hold in appropriate Banach spaces, see [8]. The arguments there were based on a specific property of these subdifferentials with respect to exact inf-convolutions of two functions. Such an argument is avoided here: Formula (1) is valid in a Banach space X as soon as the subdifferential satisfies the Separation Principle in this space.

3 First-Order Tests for Optimality

The following lemma comes from our paper [8]. It establishes a mean value inequality using the directional derivative. For the sake of completeness, we recall the simple proof.

Lemma 3.1 *Let X be a Hausdorff locally convex space, $f : X \rightarrow]-\infty, +\infty]$ be lower semicontinuous, $\bar{x} \in X$ and $x \in \text{dom } f$. Then, for every real number $\lambda \leq f(\bar{x}) - f(x)$, there exist $t_0 \in [0, 1[$ and $x_0 := x + t_0(\bar{x} - x) \in [x, \bar{x}[$ such that $\lambda \leq f'(x_0; \bar{x} - x)$ and $f(x_0) \leq f(x) + t_0\lambda$.*

Proof. Let $\lambda \in \mathbb{R}$ such that $\lambda \leq f(\bar{x}) - f(x)$ and define $g : [0, 1] \rightarrow]-\infty, +\infty]$ by $g(t) := f(x + t(\bar{x} - x)) - t\lambda$. Then g is lower semicontinuous on the compact $[0, 1]$ and $g(0) = f(x) \leq f(\bar{x}) - \lambda = g(1)$. Hence g attains its minimum on $[0, 1]$ at a point $t_0 \neq 1$. Let $x_0 := x + t_0(\bar{x} - x) \in [x, \bar{x}[$. Then, $f(x_0) - t_0\lambda = g(t_0) \leq g(0) = f(x)$ and, since $g(t_0 + t) \geq g(t_0)$ for every $t \in]0, 1 - t_0]$, we derive that

$$\forall t \in]0, 1 - t_0], \quad \frac{f(x_0 + t(\bar{x} - x)) - f(x_0)}{t} \geq \lambda.$$

Passing to the lower limit as $t \searrow 0$, we get $f'(x_0; \bar{x} - x) \geq \lambda$. The proof is complete. \blacksquare

We easily derive from Lemma 3.1 a first-order necessary and sufficient condition for optimality of lower semicontinuous functions in terms of the directional derivative:

Proposition 3.2 *Let X be a Banach space, $f : X \rightarrow]-\infty, +\infty]$ be lower semicontinuous, $U \subset X$ open convex and $\bar{x} \in U$. Then, the following are equivalent:*

- (a) $f(\bar{x}) \leq f(y)$ for every $y \in U$,
- (b) $f(\bar{x}) \leq f(y)$ for every $y \in U$ such that $f'(y; \bar{x} - y) > 0$.

Proof. Obviously, (a) implies (b). We prove that $\neg(a)$ implies $\neg(b)$. Assume there exists $x \in U$ such that $f(\bar{x}) > f(x)$. We must show that there exists $x_0 \in U$ such that $f(\bar{x}) > f(x_0)$ and $f'(x_0; \bar{x} - x_0) > 0$. Applying Lemma 3.1 with $0 < \lambda < f(\bar{x}) - f(x)$, we derive that there exist $x_0 \in [x, \bar{x}[\subset U$ and $t_0 \in [0, 1[$ such that

$$f'(x_0; \bar{x} - x) \geq \lambda > 0 \quad \text{and} \quad f(x_0) \leq f(x) + t_0 \lambda.$$

Since $f(x) + \lambda < f(\bar{x})$, $\lambda > 0$ and $t_0 \in [0, 1[$, we have $f(x_0) < f(\bar{x})$, and, since $\bar{x} - x_0 = t(\bar{x} - x)$ for some $t > 0$, we have $f'(x_0; \bar{x} - x_0) = t f'(x_0; \bar{x} - x) > 0$. \blacksquare

The following first-order sufficient condition is a straightforward consequence. It clearly contains the result quoted in the introduction. We refer to [5] for a characterization of the solution set of Minty variational inequalities governed by directional derivatives.

Corollary 3.2.1 *Let X be a Banach space, $f : X \rightarrow]-\infty, +\infty]$ be lower semicontinuous, $U \subset X$ open convex and $\bar{x} \in U$. Then:*

$$\forall y \in U, f'(y; \bar{x} - y) \leq 0 \implies \forall y \in U, f(\bar{x}) \leq f(y).$$

Now, coming back to our objective, we combine Lemma 3.1 with Theorem 2.1 to establish a first-order necessary and sufficient condition for optimality of lower semicontinuous functions in terms of any subdifferential satisfying the Separation Principle. This complements our previous result [8, Theorem 4.2] which concerned only the elementary subdifferentials.

Theorem 3.3 *Let X be a Banach space, $f : X \rightarrow]-\infty, +\infty]$ be lower semicontinuous, $U \subset X$ open convex and $\bar{x} \in U$. Then, the following are equivalent:*

- (a) $f(\bar{x}) \leq f(y)$ for every $y \in U$,
- (b) $f(\bar{x}) \leq f(y)$ for every $y \in U$ such that $\sup \langle \partial f(y), \bar{x} - y \rangle > 0$.

Proof. Obviously, (a) implies (b). Conversely, we show that $\neg(a)$ implies $\neg(b)$. We know from Proposition 3.2 that $\neg(a)$ implies the existence of $x_0 \in U$ such that $f(\bar{x}) > f(x_0)$ and $f'(x_0; \bar{x} - x_0) > 0$. Let $\varepsilon > 0$ such that $x_0 + \varepsilon B_X \subset U$ and $f(x_0) + \varepsilon < f(\bar{x})$, and apply Formula (1) at point x_0 and direction $d = \bar{x} - x_0$. We obtain a pair $(y_\varepsilon, y_\varepsilon^*) \in \partial f$ such that

$$\|y_\varepsilon - x_0\| < \varepsilon, \quad f(y_\varepsilon) < f(x_0) + \varepsilon, \quad \langle y_\varepsilon^*, y_\varepsilon - x_0 \rangle < \varepsilon, \quad \langle y_\varepsilon^*, \bar{x} - x_0 \rangle > 0,$$

from which we derive that $y_\varepsilon \in U$, $f(y_\varepsilon) < f(\bar{x})$ and $\langle y_\varepsilon^*, \bar{x} - y_\varepsilon \rangle > 0$. \blacksquare

As above, the following first-order sufficient condition is a straightforward consequence:

Corollary 3.3.1 *Let X be a Banach space, $f : X \rightarrow]-\infty, +\infty]$ be lower semicontinuous, $U \subset X$ open convex and $\bar{x} \in U$. Then:*

$$\forall y \in U, \sup \langle \partial f(y), \bar{x} - y \rangle \leq 0 \implies \forall y \in U, f(\bar{x}) \leq f(y). \quad (12)$$

We recall from [8] an interpretation of the sufficient condition (12) in terms of the monotonic behaviour of the subdifferential operator. Given a set-valued operator $T : X \rightrightarrows X^*$, or graph $T \subset X \times X^*$, we let

$$T^0 := \{(x, x^*) : \langle y^* - x^*, y - x \rangle \geq 0 \quad \forall (y, y^*) \in T\}$$

be the set of all pairs $(x, x^*) \in X \times X^*$ *monotonically related* to T . The operator T is said to be *monotone* provided $T \subset T^0$, *monotone absorbing* provided $T^0 \subset T$, *maximal monotone* provided $T = T^0$.

Theorem 3.4 *Let X be a Banach space and let $f : X \rightarrow]-\infty, +\infty]$ be lower semicontinuous. Then, the operator $\partial f : X \rightrightarrows X^*$ is monotone absorbing. In particular, for convex f , ∂f is maximal monotone.*

Proof. Let $(x, x^*) \in (\partial f)^0$. Then, $\langle y^* - x^*, y - x \rangle \geq 0$ for every $(y, y^*) \in \partial f$. Since $\partial(f - x^*)(y) = \partial f(y) - x^*$, this can be written as

$$\forall (y, y^*) \in \partial(f - x^*), \quad \langle y^*, y - x \rangle \geq 0,$$

so, by Corollary 3.3.1, $(f - x^*)(x) \leq (f - x^*)(y)$ for every $y \in X$. We then conclude from the Separation Principle that $x^* \in \partial f(x)$. Thus, $(\partial f)^0 \subset \partial f$.

For convex f , ∂f is monotone, hence maximal monotone from what precedes. ■

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